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On The Special Curves in Minkowski 4 Spacetime

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Abstract

In [1], we gave a method for constructing Bertrand curves from the spherical curves in 3 dimensional Minkowski space. In this work, we construct the Bertrand curves corresponding to a spacelike geodesic and a null helix in Minkowski 4 spacetime.

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1 Preliminary Notes

In this section, we give basic notions of spacelike and null curves in Minkowski 4-space (see [2], [3] and [6]). Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. For any vectors $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4)$ in \mathbb{R}^4 , the pseudo scalar product of x and y is defined to be $\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle, \rangle)$ a Minkowski 4-space. We write \mathbb{R}_1^4 instead of $(\mathbb{R}^4, \langle, \rangle)$. We say that a non-zero vector $x \in \mathbb{R}_1^4$ is spacelike, lightlike (null) or timelike if $\langle x, x \rangle > 0$, $\langle x, x \rangle = 0$ or $\langle x, x \rangle < 0$ respectively. The norm of the vector $x \in \mathbb{R}_1^4$ is defined by $\|x\| = \sqrt{|\langle x, x \rangle|}$. For a vector $v \in \mathbb{R}_1^4$ and a real number c , we define a hyperplane with pseudo normal v by

$HP(v, c) = \{x \in \mathbb{R}_1^4 : \langle x, v \rangle = c\}$. We call $HP(v, c)$ a spacelike hyperplane, a timelike hyperplane or a lightlike hyperplane if v is timelike, spacelike or lightlike respectively. We also define de Sitter 3-space by $S_1^3 = \{x \in \mathbb{R}_1^4 : \langle x, x \rangle = 1\}$. For any $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4), z = (z_1, z_2, z_3, z_4)$ in \mathbb{R}_1^4 , we define a vector

$$x \wedge y \wedge z = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

where (e_1, e_2, e_3, e_4) is the canonical basis of \mathbb{R}_1^4 . We can easily show that $\langle a, (x \wedge y \wedge z) \rangle = \det(a, x, y, z)$.

Let $\gamma : I \rightarrow S_1^3$ be a regular curve. We say that a regular curve γ is spacelike, timelike or null respectively, if $\gamma'(t)$ is spacelike, timelike or null at any $t \in I$, where $\gamma' = d\gamma/dt$. Now we describe the explicit differential geometry on spacelike and null curves in S_1^3 .

Let γ be a spacelike regular curve, we can reparametrise γ by the arclength $s = s(t)$. Hence, we may assume that $\gamma(s)$ is a unit speed curve. So we have the tangent vector $t(s) = \gamma'(s)$ with $\|t(s)\| = 1$. In the case when $\langle t'(s), t'(s) \rangle \neq 1$, we have a unit vector $n(s) = \frac{t'(s) - \gamma(s)}{\|t'(s) - \gamma(s)\|}$. Moreover, define $e(s) = \gamma(s) \wedge t(s) \wedge n(s)$, then we have a pseudo orthonormal frame $\{\gamma(s), t(s), n(s), e(s)\}$ of \mathbb{R}_1^4 along γ . By the standard arguments, we can show the following Frenet-Serret type formulae: Under the assumption that $\langle t'(s), t'(s) \rangle \neq 1$,

$$\begin{aligned} \gamma'(s) &= t(s) \\ t'(s) &= -\gamma(s) + \kappa_g(s) n(s) \\ n'(s) &= \kappa_g(s) \delta(\gamma(s)) t(s) + \tau_g(s) e(s) \\ e'(s) &= \tau_g(s) n(s) \end{aligned} \tag{1}$$

where $\delta(\gamma(s)) = -\text{sign}(n(s))$,

$$\begin{aligned} \kappa_g(s) &= \|t'(s) + \gamma(s)\| \\ \tau_g(s) &= \frac{\delta(\gamma(s))}{\kappa_g^2(s)} \det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s)) \end{aligned}$$

Now let $\gamma : I \rightarrow S_1^3$ be a null curve. We will assume, in the sequel, that the null curve we consider has no points at which the acceleration vector is null. Hence $\langle \gamma''(t), \gamma''(t) \rangle$ is never zero. We say that a null curve $\gamma(t)$ in \mathbb{R}_1^4 is parametrized by the pseudo-arc if $\langle \gamma''(t), \gamma''(t) \rangle = 1$. If a null curve satisfies $\langle \gamma''(t), \gamma''(t) \rangle \neq 0$, then $\langle \gamma''(t), \gamma''(t) \rangle > 0$, and

$$u(t) = \int_{t_0}^t \langle \gamma''(t), \gamma''(t) \rangle^{1/4} dt$$

becomes the pseudo-arc parameter.

A null curve $\gamma(t)$ in \mathbb{R}_1^4 with $\langle \gamma''(t), \gamma''(t) \rangle \neq 0$ is a Cartan curve if $\{\gamma'(t), \gamma''(t), \gamma'''(t)\}$ is linearly independent for any t . For a Cartan curve $\gamma(t)$ in \mathbb{R}_1^4 with pseudo-arc parameter t , there exists a pseudo orthonormal basis $\{L, N, W_1, W_2\}$ such that

$$\begin{aligned} L &= \gamma' \\ L' &= W_1 \\ N' &= -\gamma + k_1 W_1 + k_2 W_2 \\ W_1' &= -k_1 L - N \\ W_2' &= -k_2 L \end{aligned} \tag{2}$$

where $\langle L, N \rangle = 1, \langle L, W_1 \rangle = \langle L, W_2 \rangle = \langle N, W_1 \rangle = \langle N, W_2 \rangle = \langle W_1, W_2 \rangle = 0$. We call $\{L, N, W_1, W_2\}$ as the Cartan frame and $\{k_1, k_2\}$ as the Cartan curvatures of γ . Since the Cartan frame is unique up to orientation, the number of the Cartan curvatures is minimum and the Cartan curvatures are invariant under Lorentz transformations, the set $\{L, N, W_1, W_2, k_1, k_2\}$ corresponds to the Frenet apparatus of a space curve. A direct computation shows that the values of the Cartan curvatures are

$$\begin{aligned} k_1 &= \frac{1}{2a^2} (\langle \gamma''', \gamma''' \rangle + 2aa'' - 4(a')^2) \\ k_2 &= -\frac{1}{a^4} \det(\gamma', \gamma'', \gamma''', \gamma^{(4)}) \end{aligned} \tag{3}$$

Theorem 1.1 *Let $\gamma(t)$ in \mathbb{R}_1^4 be a Cartan curve. Then γ is a pseudo-spherical curve iff k_2 is a nonzero constant.*

Theorem 1.2 *A Cartan curve $\gamma(t)$ in \mathbb{R}_1^4 fully lies on a pseudo-sphere iff there exists a fixed point A such that for each $t \in I$, $\langle A - \gamma(t), \gamma'(t) \rangle = 0$.*

2 Bertrand Curve Corresponding to A Space-like Geodesic on S_1^3

Theorem 2.1 *Let γ be a spacelike geodesic curve on S_1^3 . Then,*

$$\tilde{\gamma}(s) = a \int \gamma(v) dv + a \coth \theta \int e(v) dv + c$$

is a Bertrand curve where a and θ are constant numbers, c is a constant vector.

Proof. We will use the frame $\{\gamma(s), t(s), n(s), e(s)\}$ of γ given in the previous section. In this frame, let we choose $e(s)$ as a timelike vector (If $e(s)$ is a spacelike vector, the proof is similar). Hence $n(s)$ is spacelike and $\delta(\gamma(s)) = -1$. Using the equation (1), we can easily calculate that

$$\begin{aligned}\tilde{\gamma}'(s) &= a[\gamma(s) + \coth \theta e(s)] \\ \tilde{\gamma}''(s) &= a[t(s) + \coth \theta \tau_g(s) n(s)] \\ \tilde{\gamma}'''(s) &= a[-\gamma(s) + \delta(\gamma(s)) \kappa_g(s) \tau_g(s) t(s) \\ &\quad + (\kappa_g(s) + \coth \theta \tau_g'(s)) n(s) + \coth \theta \tau_g^2(s) e(s)]\end{aligned}$$

Since $\langle \tilde{\gamma}'(s), \tilde{\gamma}'(s) \rangle = -\frac{a^2}{\sinh^2 \theta}$, the curve $\tilde{\gamma}$ is timelike. If we calculate the first and second curvatures of $\tilde{\gamma}$ by using the equations in [8], we have

$$\begin{aligned}\kappa(s) &= \frac{\sinh^2 \theta \sqrt{1 + \coth^2 \theta \tau_g^2}}{a} \\ \tau(s) &= \frac{A \sinh \theta}{a \sqrt{1 + \coth^2 \theta \tau_g^2}}\end{aligned}$$

where $A = \sqrt{\cosh^2 \theta (\tau_g^2 + 1)^2 - \kappa_g^2 (1 + \coth^2 \theta \tau_g^2)}$. Since τ_g and κ_g are constants, we can choose $\beta = \frac{-a \sinh \theta \sqrt{1 + \coth^2 \theta \tau_g^2}}{A}$ and $\alpha = \frac{a \coth^2 \theta}{\sqrt{1 + \coth^2 \theta \tau_g^2}}$, then we have $\alpha \kappa + \beta \tau = 1$. Hence $\tilde{\gamma}$ is a Bertrand curve.

3 Bertrand Curve Corresponding to A Null Helix on S_1^3

Theorem 3.1 *Let γ be a null helix on S_1^3 . Then,*

$$\tilde{\gamma}(s) = a \int L(v) dv + a \coth \theta \int W_2(v) dv + c$$

is a Bertrand curve where a and θ are constant numbers, c is a constant vector.

Proof.

$$\begin{aligned}\tilde{\gamma}'(t) &= a[L(s) + \coth \theta W_2(t)] \\ \tilde{\gamma}''(t) &= a[1 - \coth \theta k_2] W_1(t) \\ \tilde{\gamma}'''(t) &= a[k_1 (\coth \theta - 1) L(t) - (1 - \coth \theta k_2) N(t)]\end{aligned}$$

Since $\langle \tilde{\gamma}'(t), \tilde{\gamma}'(t) \rangle = a^2 \coth^2 \theta$, the curve $\tilde{\gamma}$ is spacelike. If we calculate the first and second curvatures of $\tilde{\gamma}$, we have

$$\begin{aligned}\kappa(t) &= \frac{(1 - \coth \theta k_2)}{a \coth^2 \theta} \\ \tau(t) &= \frac{\sqrt{k_1^2 \cosh^2 \theta - 1}}{\cosh \theta}\end{aligned}$$

Since k_1 and k_2 are constants, we can choose $\beta = -\frac{\cosh^3 \theta}{\sqrt{k_1^2 \cosh^2 \theta - 1}}$ and $\alpha = \frac{a \cosh^2 \theta}{(1 - \coth \theta k_2)}$, then we have $\alpha\kappa + \beta\tau = 1$. Hence $\tilde{\gamma}$ is a Bertrand curve.

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